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LETTER TO THE EDITOR

Extremal trajectories for stochastic equations obtained directly from the Langevin differential operator

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Abstract. We show that the differential operator for the extremal trajectory of a stochastic process can be connected directly to the systematic part of the differential operator that defines the stochastic equation. By assuming linearity in this operator and Gaussianity for the fluctuation, we are able to write these relations for Markovian as well as non-Markovian processes.

According to the point of view of the path-integral formalism, if the integrals involved are Gaussian then the conditional probability density can be expressed as an exponential of a functional $S[x(t)]$ evaluated along an extremal path. The extremal path is, among all the possible paths that connect the fixed end points, that one for which S has an extremum value. The calculation of this extremal path usually requires a variational procedure that gives us a differential equation for that path.

Even if the path integrals involved are not Gaussian, as occurs in the case of complicated potentials, the extremal path can be useful in many calculations; some of them are good approximations [1–4].

In order to find the extremal trajectory, here we start with a general stochastic equation with additive and Gaussian noise

$$D(x)=f(t) \tag{1}$$

where we consider D as a linear differential operator, which hereafter is referred to as the Langevin operator and $f(t)$ a stochastic function that can be Markovian or not. Then we make a variational calculation starting from this equation and we are able to show that the extremal path differential operator can be written as a product of D, its adjoint D^* and other factors if the process is non-Markovian. In those cases where the operator D has time-dependent coefficients, this factorization can be useful to solve a complicated extremal path differential equation.

We start from equation (1) and as f is Gaussian we define the action as the functional [5]

$$S[x(t)]=\int_{t_1}^{t_2} L(t)\,\mathrm{d}t \tag{2}$$

where

$$L(t)=(D(x))^2 \tag{3}$$

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and

$$D = a_0 \frac{d^{(N)}}{dt^N} + a_1 \frac{d^{(N-1)}}{dt^{N-1}} + \ldots + a_N \tag{4}$$

with a_j analytic functions of t.

Applying the stationary condition $\delta S = 0$ to equation (2), and after several integrations by parts one obtains

$$\sum_{k=0}^{N} (-1)^k \frac{d^{(k)}}{dt^k} \frac{\partial L}{\partial x^{(k)}} = 0 \tag{5}$$

which is the Lagrange differential equation for the extremal path.

Now, combining equations (3), (4) and (5), we have

$$\frac{\partial L}{\partial x^{(k)}} = 2a_{N-k} D_x \tag{6}$$

$$\sum_{k=0}^{N} (-1)^k \frac{d^{(k)}}{dt^k} a_{N-k} D_x = 0 \tag{7}$$

and we identify one of the factors in this equation as the adjoint of D [6]

$$D^* = \sum_{k=0}^{N} (-1)^k \frac{d^{(k)}}{dt^k} a_{N-k}. \tag{8}$$

Then we can write the differential equation (5) in the compact form

$$D^* Dx = 0 \tag{9}$$

so we observe that in the Markovian case the extremal differential operator is the product of the Langevin operator times its adjoint. If the coefficients of D are constants then D and D^* commute and in this case the general solution for the extremal path is a linear combination of the solutions of $D(x) = 0$ and $D^*(x) = 0$. If not all a_j are constants then D and D^* do not commute and the solutions of $D^*(x) = 0$ are in general not solutions of $D^* D(x) = 0$. We also notice that the operator $D^* D$ is self-adjoint.

Now we go back to equation (1)

$$Dx = f(t) \tag{10}$$

and we assume a correlation function for f as

$$\langle f(t_1) f(t_2) \rangle = (D/\tau) \exp(-|t_1 - t_2|/\tau) \tag{11}$$

where τ is the correlation time; then we can write, together with equation (10), the equation

$$\tau \frac{df}{dt} = -f + \xi \tag{12}$$

with ξ being delta correlated.

Taking the derivative of $D(x)$ we obtain

$$\left(1 + \tau \frac{d}{dt}\right) Dx = \xi(t). \tag{13}$$

Then as ξ is Gaussian and delta correlated we make a similar calculation as we did in the first case

$$L(t)=\xi^2=\left[\left(1+\tau\frac{\mathrm{d}}{\mathrm{d}t}\right)Dx\right]^2 \tag{14}$$

where D is already defined in equation (4).

Taking the stationary condition $\delta S=0$ we have

$$\sum_{k=0}^{N+1}(-1)^k\frac{\mathrm{d}^{(k)}}{\mathrm{d}t^k}\frac{\partial L}{\partial x^{(k)}}=0 \tag{15}$$

and after doing the calculations we can write the equation for the extremal path as

$$\left[\tau D_t^*+\left(1-\tau\frac{\mathrm{d}}{\mathrm{d}t}\right)D^*\right]\left(1+\tau\frac{\mathrm{d}}{\mathrm{d}t}\right)Dx=0 \tag{16}$$

where we define the operator D_t^* as

$$D_t^*=\sum_{k=0}^{N}(-1)^k\frac{\mathrm{d}^{(k)}}{\mathrm{d}t^k}\dot{a}_{N-k}. \tag{17}$$

So equation (16) is a general and exact expression for the extremal path differential equation and it is interesting to have it factorized in this form.

First we notice that these factors do not commute (if the coefficients of D are not constants); we see that any solution of $D(x)=0$ (which is the Langevin systematic part) is a particular solution for the extremal and, also, any solution of the equation $(1+\tau(\mathrm{d}/\mathrm{d}t))D(x)=0$ is a particular solution of the extremal; this last equation is exactly the systematic part of equation (13). We continue taking factors to the left of equation (16), and if one is able to solve each time the differential equation of the factor in the right side, then we are lowering the order of the external differential equation we want to solve. We point out that this equation can have variable coefficients. We also notice in equation (16) that if the correlation time τ goes to zero then it reduces to equation (9) which is the Markovian limit.

If the operator D defined in equation (4) has constant coefficients, then equation (16) can be written as

$$\left(1-\tau^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)D^*Dx=M^*MD^*Dx=0 \tag{18}$$

where we are defining a *memory* operator $M=1+\tau(\mathrm{d}/\mathrm{d}t)$ and its adjoint $M^*=1-\tau(\mathrm{d}/\mathrm{d}t)$.

We observe that the external operator in this case is self-adjoint, and its factors commute; therefore we can write the solution for the external path as a linear combination of the solutions of $D(x)=0$, $D^*(x)=0$ and the functions $\exp(t/\tau)$ and $\exp(-t/\tau)$. These exponentials are the solutions that correspond to the memory operators and we see that they give to the external path the memory contribution, no matter the precise form of the Langevin operator D.

Looking at equations (9) and (18), we observe in both of them a kind of 'square' operators D^*D and M^*M. This 'square' of course depends on the Gaussianity of noise and it is interesting to see that these products of operators involve or are equivalent to a variational process.

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